10-2 Parabolas

Main Ideas

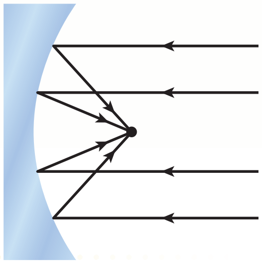
- Write equations of parabolas in standard form.
- Graph parabolas.

New Vocabulary

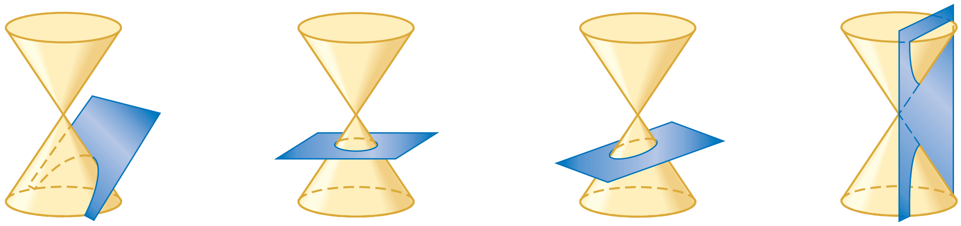
parabola
conic section
focus
directrix
latus rectum

GET READY for the Lesson

A mirror or other reflective object in the shape of a parabola reflects all parallel incoming rays to the same point. Or, if that point is the source of rays, the reflected rays are all parallel.



Equations of Parabolas In Chapter 5, you learned that the graph of an equation of the form $y = ax^2 + bx + c$ is a **parabola**. A parabola can also be obtained by slicing a double cone on a slant as shown below on the left. Any figure that can be obtained by slicing a double cone is called a **conic section**. Other conic sections are also shown below.

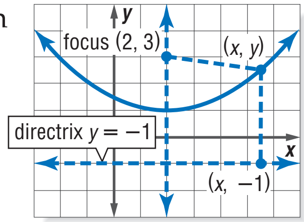


Study Tip

Focus of a Parabola

The focus is the special point referred to at the beginning of the lesson.

A parabola can also be defined as the set of all points in a plane that are the same distance from a given point called the **focus** and a given line called the **directrix**. The parabola at the right has its focus at (2, 3), and the equation of its directrix is $y = -1$. You can use the Distance Formula to find an equation of this parabola.



Let (x, y) be any point on this parabola. The distance from this point to the focus must be the same as the distance from this point to the directrix. The distance from a point to a line is measured along the perpendicular from the point to the line.

distance from (x, y) to $(2, 3)$ = distance from (x, y) to $(x, -1)$

$$\sqrt{(x-2)^2 + (y-3)^2} = \sqrt{(x-x)^2 + [y-(-1)]^2}$$

$$(x-2)^2 + (y-3)^2 = 0^2 + (y+1)^2 \quad \text{Square each side.}$$

$$(x-2)^2 + y^2 - 6y + 9 = y^2 + 2y + 1 \quad \text{Square } y - 3 \text{ and } y + 1.$$

$$(x-2)^2 + 8 = 8y \quad \text{Isolate the } y\text{-terms.}$$

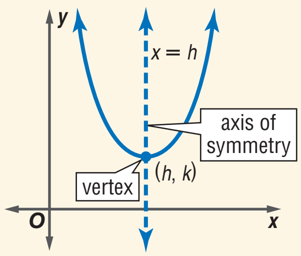
$$\frac{1}{8}(x-2)^2 + 1 = y \quad \text{Divide each side by 8.}$$

An equation of the parabola with focus at (2, 3) and directrix with equation $y = -1$ is $y = \frac{1}{8}(x-2)^2 + 1$. The equation of the *axis of symmetry* for this parabola is $x = 2$. The axis of symmetry intersects the parabola at a point called the *vertex*. The vertex is the point where the graph turns. The vertex of this parabola is at (2, 1). Since $\frac{1}{8}$ is positive, the parabola opens upward. Any equation of the form $y = ax^2 + bx + c$ can be written in standard form.

KEY CONCEPT — *Equation of a Parabola*

The standard form of the equation of a parabola with vertex (h, k) and axis of symmetry $x = h$ is $y = a(x - h)^2 + k$.

- If $a > 0$, k is the minimum value of the related function and the parabola opens upward.
- If $a < 0$, k is the maximum value of the related function and the parabola opens downward.



EXAMPLE — Analyze the Equation of a Parabola

1 **Write $y = 3x^2 + 24x + 50$ in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.**

$y = 3x^2 + 24x + 50$	Original equation
$= 3(x^2 + 8x) + 50$	Factor 3 from the x-terms.
$= 3(x^2 + 8x + ■) + 50 - 3(■)$	Complete the square on the right side.
$= 3(x^2 + 8x + 16) + 50 - 3(16)$	The 16 added when you complete the square is multiplied by 3.
$= 3(x + 4)^2 + 2$	
$= 3[x - (-4)]^2 + 2$	$(h, k) = (-4, 2)$

The vertex of this parabola is located at $(-4, 2)$, and the equation of the axis of symmetry is $x = -4$. The parabola opens upward.

Study Tip

Look Back

To review **completing the square,** see Lesson 5–5.

CHECK Your Progress

1. Write $y = 4x^2 + 16x + 34$ in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.

Study Tip

Translations

If h is positive, translate the graph h units to the right. If h is negative, translate the graph h units to the left. Similarly, if k is positive, translate the graph k units up. If k is negative, translate the graph k units down.

Graph Parabolas You can use symmetry and translations to graph parabolas. The equation $y = a(x - h)^2 + k$ can be obtained from $y = ax^2$ by replacing x with $x - h$ and y with $y - k$. Therefore, the graph of $y = a(x - h)^2 + k$ is the graph of the parent function $y = ax^2$ translated h units to the right or left and k units up or down.

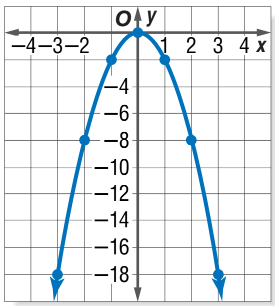
EXAMPLE — Graph Parabolas

2 **Graph each equation.**

a. $y = -2x^2$

For this equation, $h = 0$ and $k = 0$. The vertex is at the origin. Since the equation of the axis of symmetry is $x = 0$, substitute some small positive integers for x and find the corresponding y-values.

x	y
1	−2
2	−8
3	−18

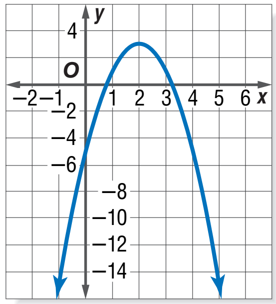


Since the graph is symmetric about the y-axis, the points at $(-1, -2)$, $(-2, -8)$, and $(-3, -18)$ are also on the parabola. Use all of these points to draw the graph.

Notice that each side of the graph is the reflection of the other side about the y-axis.

b. $y = -2(x - 2)^2 + 3$

The equation is of the form $y = a(x - h)^2 + k$, where $h = 2$ and $k = 3$. The graph of this equation is the graph of $y = -2x^2$ in part **a** translated 2 units to the right and up 3 units. The vertex is now at (2, 3).



CHECK Your Progress

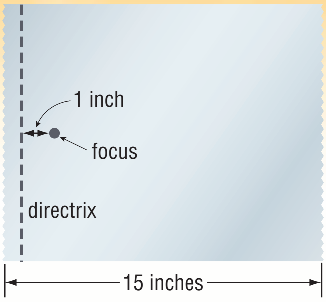
2A. $y = 3x^2$

2B. $y = 3(x - 1)^2 - 4$

ALGEBRA LAB

Parabolas

Step 1 Start with a sheet of wax paper that is about 15 inches long and 12 inches wide. Make a line that is perpendicular to the sides of the sheet by folding the sheet near one end. Open up the paper again. This line is the directrix. Mark a point about midway between the sides of the sheet so that the distance from the directrix is about 1 inch. This is the focus.



Put the focus on top of any point on the directrix and crease the paper. Make about 20 more creases by placing the focus on top of other points on the directrix. The lines form the outline of a parabola.

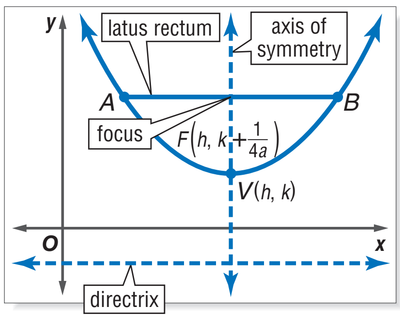
Step 2 Start with a new sheet of wax paper. Form another outline of a parabola with a focus that is about 3 inches from the directrix.

Step 3 On a new sheet of wax paper, form a third outline of a parabola with a focus that is about 5 inches from the directrix.

ANALYZE THE RESULTS

Compare the shapes of the three parabolas. How does the distance between the focus and the directrix affect the shape of a parabola?

The shape of a parabola and the distance between the focus and directrix depend on the value of a in the equation. The line segment through the focus of a parabola and perpendicular to the axis of symmetry is called the **latus rectum.** The endpoints of the latus rectum lie on the parabola.



In the figure, the latus rectum is $\overline{AB}$. The length of the latus rectum of the parabola with equation $y = a(x - h)^2 + k$ is $\left|\frac{1}{a}\right|$ units. The endpoints of the latus rectum are $\left|\frac{1}{2a}\right|$ units from the focus.

Equations of parabolas with vertical axes of symmetry have the parent function $y = x^2$ and are of the form $y = a(x - h)^2 + k$. These are functions. Equations of parabolas with horizontal axes of symmetry are of the form $x = a(y - k)^2 + h$ and are not functions. The parent graph for these equations is $x = y^2$.

KEY CONCEPT — Information About Parabolas

Form of Equation	$y = a(x - h)^2 + k$	$x = a(y - k)^2 + h$
Vertex	(h, k)	(h, k)
Axis of Symmetry	$x = h$	$y = k$
Focus	$\left(h, k + \frac{1}{4a}\right)$	$\left(h + \frac{1}{4a}, k\right)$
Directrix	$y = k - \frac{1}{4a}$	$x = h - \frac{1}{4a}$
Direction of Opening	upward if $a > 0$, downward if $a < 0$	right if $a > 0$, left if $a < 0$
Length of Latus Rectum	$\left\lvert\frac{1}{a}\right\rvert$ units	$\left\lvert\frac{1}{a}\right\rvert$ units

EXAMPLE Graph an Equation Not in Standard Form

Study Tip

When graphing these functions, it may be helpful to sketch the graph of the parent function.

3 **Graph $4x - y^2 = 2y + 13$.**

First, write the equation in the form $x = a(y - k)^2 + h$.

$4x - y^2 = 2y + 13$	There is a y^2 term, so isolate the y and y^2 terms.
$4x = y^2 + 2y + 13$	Add y^2 to each side.
$4x = (y^2 + 2y + ■) + 13 - ■$	Complete the square.
$4x = (y^2 + 2y + 1) + 13 - 1$	Add and subtract 1, since $\left(\frac{2}{2}\right)^2 = 1$.
$4x = (y + 1)^2 + 12$	Write $y^2 + 2y + 1$ as a square.
$x = \frac{1}{4}(y + 1)^2 + 3$	$(h, k) = (3, -1)$

Then use the following information to draw the graph based on the parent graph, $x = y^2$.

vertex: $(3, -1)$

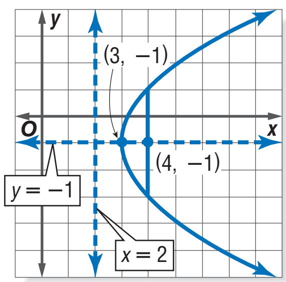
axis of symmetry: $y = -1$

focus: $\left(3 + \frac{1}{4\left(\frac{1}{4}\right)}, -1\right)$ or $(4, -1)$

directrix: $x = 3 - \frac{1}{4\left(\frac{1}{4}\right)}$ or 2

direction of opening: right, since $a > 0$

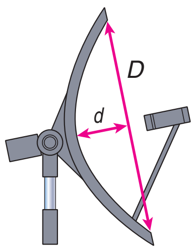
length of latus rectum: $\left\lvert\frac{1}{\left(\frac{1}{4}\right)}\right\rvert$ or 4 units



The graph is wider than the graph of $x = y^2$ since $a < 1$ and shifted 3 units right and 1 unit down.

CHECK Your Progress Graph each equation.

3A. $3x - y^2 = 4y + 25$ **3B.** $y = x^2 + 6x - 4$



Real-World Link

The important characteristics of a satellite dish are the diameter D, depth d, and the ratio $\frac{f}{D}$, where f is the distance between the focus and the vertex. A typical dish has the values $D = 60$ cm, $d = 6.25$ cm, and $\frac{f}{D} = 0.6$.

Source: 2000networks.com

EXAMPLE Write and Graph an Equation for a Parabola

4 **SATELLITE TV** **Use the information at the left about satellite dishes.**

a. Write an equation that models a cross section of a satellite dish. Assume that the focus is at the origin and the parabola opens to the right.

First, solve for f. Since $\frac{f}{D} = 0.6$, and $D = 60$, $f = 0.6(60)$ or 36.

The focus is at (0, 0), and the parabola opens to the right. So the vertex must be at (−36, 0). Thus, $h = -36$ and $k = 0$. Now find a.

$-36 + \frac{1}{4a} = 0$ — $h = -36$; The x-coordinate of the focus is 0.

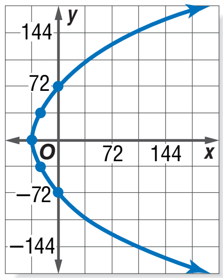
$\frac{1}{4a} = 36$ — Add 36 to each side.

$1 = 144a$ — Multiply each side by $4a$.

$\frac{1}{144} = a$ — Divide each side by 144.

An equation of the parabola is $x = \frac{1}{144}y^2 - 36$.

b. Graph the equation.



The length of the latus rectum is $\left|\frac{1}{\frac{1}{144}}\right|$ or 144 units, so the graph must pass through (0, 72) and (0, −72). According to the diameter and depth of the dish, the graph must pass through (−29.75, 30) and (−29.75, −30). Use these points and the information from part **a** to draw the graph.

CHECK Your Progress

4. Write and graph an equation for a satellite dish with diameter D of 34 inches and ratio $\frac{f}{D}$ of 0.6.

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CHECK Your Understanding

Example 1 (p. 568)

1. Write $y = 2x^2 - 12x + 6$ in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.

Examples 2, 3 (pp. 568–570)

Graph each equation.

2. $y = (x - 3)^2 - 4$

3. $y = 2(x + 7)^2 + 3$

4. $y = -3x^2 - 8x - 6$

5. $x = \frac{2}{3}y^2 - 6y + 12$

Example 4 (pp. 570–571)

6. **COMMUNICATION** A microphone is placed at the focus of a parabolic reflector to collect sound for the television broadcast of a football game. Write an equation for the cross section, assuming that the focus is at the origin, the focus is 6 inches from the vertex, and the parabola opens to the right.

Exercises

HOMEWORK HELP

For Exercises	See Examples
7–10	1
11–14	2
15–19	3
20–23	4

Write each equation in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.

7. $y = x^2 - 6x + 11$

8. $x = y^2 + 14y + 20$

9. $y = \frac{1}{2}x^2 + 12x - 8$

10. $x = 3y^2 + 5y - 9$

Graph each equation.

11. $y = -\frac{1}{6}x^2$

12. $x = \frac{1}{2}y^2$

13. $y = \frac{1}{3}(x + 6)^2 + 3$

14. $y = -\frac{1}{2}(x - 1)^2 + 4$

15. $4(x - 2) = (y + 3)^2$

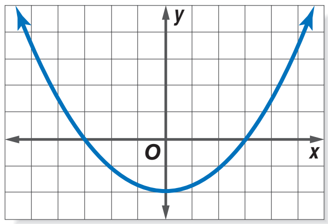
16. $(y - 8)^2 = -4(x - 4)$

17. $y = x^2 - 12x + 20$

18. $x = y^2 - 14y + 25$

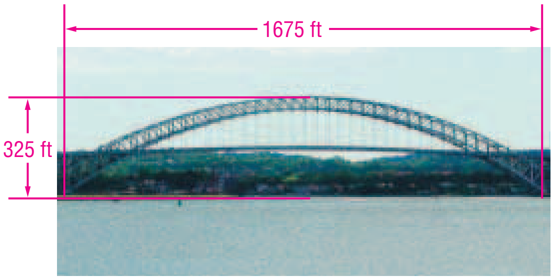
19. $x = 5y^2 + 25y + 60$

20. Write an equation for the graph at the right.



21. **MANUFACTURING** The reflective surface in a flashlight has a parabolic shape with a cross section that can be modeled by $y = \frac{1}{3}x^2$, where x and y are in centimeters. How far from the vertex should the filament of the light bulb be located?

22. **BRIDGES** The Bayonne Bridge connects Staten Island, New York, to New Jersey. It has an arch in the shape of a parabola. Write an equation of a parabola to model the arch, assuming that the origin is at the surface of the water, beneath the vertex of the arch.



23. **FOOTBALL** When a ball is thrown or kicked, the path it travels is shaped like a parabola. Suppose a football is kicked from ground level, reaches a maximum height of 25 feet, and hits the ground 100 feet from where it was kicked. Assuming that the ball was kicked at the origin, write an equation of the parabola that models the flight of the ball.

For Exercises 24–27, use the equation $x = 3y^2 + 4y + 1$.

24. Draw the graph. Find the x-intercept(s) and y-intercept(s).

25. What is the equation of the axis of symmetry?

26. What are the coordinates of the vertex?

27. How does the graph compare to the graph of the parent function $x = y^2$?

Write an equation for each parabola described below. Then draw the graph.

28. vertex (0, 1), focus (0, 5)

29. vertex (8, 6), focus (2, 6)

30. focus $(-4, -2)$, directrix $x = -8$

31. vertex (1, 7), directrix $y = 3$

32. vertex $(-7, 4)$, axis of symmetry $x = -7$, measure of latus rectum 6, $a < 0$

33. vertex (4, 3), axis of symmetry $y = 3$, measure of latus rectum 4, $a > 0$

Identify the coordinates of the vertex and focus, the equations of the axis of symmetry and directrix, and the direction of opening of the parabola with the given equation. Then find the length of the latus rectum and graph the parabola.

34. $y = 3x^2 - 24x + 50$

35. $y = -2x^2 + 5x - 10$

36. $x = -4y^2 + 6y + 2$

37. $x = 5y^2 - 10y + 9$

38. $y = \frac{1}{2}x^2 - 3x + \frac{19}{2}$

39. $x = -\frac{1}{3}y^2 - 12y + 15$

40. **UMBRELLAS** A beach umbrella has an arch in the shape of a parabola that opens downward. The umbrella spans 9 feet across and $1\frac{1}{2}$ feet high. Write an equation of a parabola to model the arch, assuming that the origin is at the point where the pole and umbrella meet, beneath the vertex of the arch.

EXTRA PRACTICE
See pages 912, 935.
Math Online
Self-Check Quiz at algebra2.com

(t)James Rooney, (b)Hisham F. Ibrahim/Getty Images

H.O.T. Problems

41. **REASONING** How do you change the equation of the parent function $y = x^2$ to shift the graph to the right?

42. **OPEN ENDED** Write an equation for a parabola that opens to the left. Use the parent graph to sketch the graph of your equation.

43. **FIND THE ERROR** Yasu is finding the standard form of the equation $y = x^2 + 6x + 4$. What mistake did she make in her work?

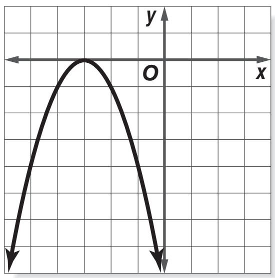
$y = x^2 + 6x + 4$
$y = x^2 + 6x + 9 + 4$
$y = (x + 3)^2 + 4$

44. **CHALLENGE** The parabola with equation $y = (x - 4)^2 + 3$ has its vertex at (4, 3) and passes through (5, 4). Find an equation of a different parabola with its vertex at (4, 3) and that passes through (5, 4).

45. ***Writing in Math*** Use the information on page 567 to explain how parabolas can be used in manufacturing. Include why a car headlight with a parabolic reflector is better than one with an unreflected light bulb.

STANDARDIZED TEST PRACTICE

46. **ACT/SAT** Which is the parent function of the graph shown below?



A $y = -x$
B $y = -\sqrt{x}$
C $y = -|x|$
D $y = -x^2$

47. **REVIEW** $\log_9 30 =$

F $\log_{10} 9 + \log_{10} 30$
G $\log_{10} 9 - \log_{10} 30$
H $(\log_{10} 9)(\log_{10} 30)$
J $\dfrac{\log_{10} 30}{\log_{10} 9}$

Spiral Review

Find the distance between each pair of points with the given coordinates. (Lesson 10-1)

48. (7, 3), (−5, 8)
49. (4, −1), (−2, 7)
50. (−3, 1), (0, 6)

51. **RADIOACTIVITY** The decay of Radon-222 can be modeled by the equation $y = ae^{-0.1813t}$, where t is measured in days. What is the half-life of Radon-222? (Lesson 9-6)

52. **HEALTH** Alisa's heart rate is usually 120 beats per minute when she runs. If she runs for 2 hours every day, about how many times will her heart beat during the amount of time she exercises in two weeks? Express in scientific notation. (Lesson 6-1)

GET READY for the Next Lesson

PREREQUISITE SKILL **Simplify each radical expression.** (Lessons 7-1 and 7-2)

53. $\sqrt{16}$
54. $\sqrt{25}$
55. $\sqrt{81}$
56. $\sqrt{144}$
57. $\sqrt{12}$
58. $\sqrt{18}$
59. $\sqrt{48}$
60. $\sqrt{72}$